

HELICES IN THE EUCLIDEAN 5-SPACE E^5

MELEK MASAL, A. ZEYNEP AZAK

ABSTRACT. In this study, we have identified V_3 slant helix (2^{nd} type slant helix, V_5 slant helix (3^{rd} type slant helix) and attained some characteristic properties in the Euclidean 5-Space E^5 . In addition to this, we have proven that there are no other helices other than V_1 helix (inclined curve), V_3 slant helix and V_5 slant helix in the 5-dimensional Euclidean space E^5 .

1. INTRODUCTION

The theory of curves is one of the fundamental topics of differential geometry. Some specific curves play important roles in a variation of sciences. As an example, the helix curves can often be seen in the fields of biology and computer technologies along with the daily life, [2]. The classic results in R^3 that are related to the helix curves and have so many fields of application, were given by M.A. Lancet in 1802 and by B. de Saint Venant in 1845, [4]. There have been many studies related to the slant helices and darboux helices in the Euclidean 3-Space, [2, 7, 8, 14] and some results have been achieved that are related to the helices and B_2 slant helix (3rd type slant helix) in the Euclidean 4-Space, [9, 11, 12]. Apart from that, while different characterizations have been given for the inclined and non-null inclined curves in the Euclidean 5-Space and Lorentzian space, [1, 3]. Moreover the non-null helices have been examined in the Lorentzian 6-Space and new characterizations have been reached for the V_n slant helix in the n -dimensional Euclidean Space, [5, 13]. In this study, V_3 slant helix and V_5 slant helix have been identified and some results have been obtained in the five dimensional Euclidean space E^5 . Then we have proven that there are no other helices other than V_1 -helix, V_3 slant helix and V_5 slant helix in E^5 .

2. PRELIMINARIES

Let $\alpha: I \subset \mathbb{R} \rightarrow E^5$ be an arbitrary curve in E^5 . Recall that the curve α is said to be of unit speed curve if $\langle \alpha'(s), \alpha'(s) \rangle = 1$ where \langle, \rangle is the standard scalar product in the Euclidean space E^5 given by

$$\langle X, Y \rangle = \sum_{i=1}^5 x_i y_i$$

for each $X = (x_1, x_2, x_3, x_4, x_5), Y = (y_1, y_2, y_3, y_4, y_5) \in E^5$. In particular, the norm of a vector X is given by $\|X\| = \sqrt{\langle X, X \rangle}$, [6]. Let $\{V_1, V_2, V_3, V_4, V_5\}$ be the moving frame along α . The Frenet equations of the curve α are given by

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$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ V_4' \\ V_5' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & 0 \\ 0 & 0 & -k_3 & 0 & k_4 \\ 0 & 0 & 0 & -k_4 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

where V_i are called i^{th} Frenet vectors and the functions k_i are called i^{th} curvatures of the curve α , [6].

A regular curve is called a W-curve if it has constant Frenet curvatures, [10].

3. V_1 HELICES IN THE FIVE DIMENSIONAL EUCLIDEAN SPACE E^5

Definition 3.1. Let $\alpha: I \subset \mathbb{R} \rightarrow E^5$ be a unit speed curve. If the tangent vector V_1 of the curve α makes a constant angle with the fixed direction U , then α is either a V_1 -helix or inclined curve, [1].

Here $\langle V_1, U \rangle = \cos \theta = \text{constant} \neq 0$ expression can be written from the definition 3.1.

Theorem 3.1. Let α be a unit speed regular curve in E^5 . Then α is a V_1 -helix if and only if the function,

$$\left(\frac{k_1}{k_2} \right)^2 + \frac{1}{k_3^2} \left[\left(\frac{k_1}{k_2} \right)' \right]^2 + \frac{1}{k_4^2} \left[\frac{k_1 k_3}{k_2} + \left[\frac{1}{k_3} \left(\frac{k_1}{k_2} \right)' \right]' \right]^2$$

is constant.

Furthermore;

$$U = \cos \theta \left[V_1 + \frac{k_1}{k_2} V_3 + \frac{1}{k_3} \left(\frac{k_1}{k_2} \right)' V_4 + \frac{1}{k_4} \left(\frac{k_1 k_3}{k_2} + \left[\frac{1}{k_3} \left(\frac{k_1}{k_2} \right)' \right]' \right) V_5 \right]$$

where θ is the angle between the vectors V_1 and U , [1].

Theorem 3.2. Let α be a unit speed regular curve in E^5 . Then, α is a V_1 -helix if and only if the following equalities is satisfied

$$k_4 f(s) = \frac{k_1 k_3}{k_2} + \left[\frac{1}{k_3} \left(\frac{k_1}{k_2} \right)' \right]'$$

and

$$\frac{1}{k_4} \frac{d}{ds} f(s) = -\frac{1}{k_3} \left(\frac{k_1}{k_2} \right)'$$

where f is C^2 -function, [1].

Theorem 3.3. α is a unit speed curve in E^5 . Then α is a V_1 -helix if and only if the equation is satisfied.

$$\frac{1}{k_3} \left(\frac{k_1}{k_2} \right)' = \left(A - \int \left[\frac{k_1 k_3}{k_2} \sin \int k_4 ds \right] ds \right) \sin \int k_4 ds - \left(B + \left[\int \frac{k_1 k_3}{k_2} \cos \int k_4 ds \right] ds \right) \cos \int k_4 ds$$

for some constant A and B , [1].

Now, we will examine the helices other than V_1 -helix in the five dimensional Euclidean space E^5 .

4. V_3 SLANT HELIX (2^{nd} TYPE SLANT HELIX) IN THE FIVE DIMENSIONAL
EUCLIDEAN SPACE E^5

Definition 4.1. Let α be a unit speed curve with nonzero curvatures k_i , ($1 \leq i \leq 4$) in E^5 . If the third unit Frenet vector field V_3 of the curve α makes a constant angle ϕ with the fixed direction U , then α is called a V_3 -slant helix (or 2^{nd} type slant helix). (Suppose that $\langle U, U \rangle = 1$).

Theorem 4.1. Let α be a unit speed curve in E^5 . Then α is a V_3 slant helix if and only if $\frac{k_2}{k_1} = \text{const } t$ and $\frac{k_3}{k_4} = \text{const } t$.

Proof: If α is a V_3 -slant helix and U is a fixed unit vector, then the following equality can be written as

$$(4.1) \quad \langle V_3, U \rangle = \cos \phi = \text{const } t \neq 0.$$

Taking the differential of equation (4.1) with respect to s and using the Frenet equations, we obtain

$$\langle -k_2 V_2 + k_3 V_4, U \rangle = 0.$$

Therefore, U lies on the hyperplane spanned by the Frenet vectors V_1, V_3 and V_5 . Then, we reach

$$(4.2) \quad U = u_1 V_1 + u_3 V_3 + u_5 V_5$$

where, $u_i = u_i(s)$ and $u_3 = \cos \phi = \text{const } t$.

The differential of equation (4.2), we have

$$u'_1 V_1 + (u_1 k_1 - u_3 k_2) V_2 + u'_3 V_3 + (u_3 k_3 - u_5 k_4) V_4 + u'_5 V_5 = 0.$$

By the above equality, the coefficients V_i are zero for $1 \leq i \leq 5$. So we get

$$\begin{aligned} u'_1 &= 0 \\ u_1 k_1 - u_3 k_2 &= 0 \\ u'_3 &= 0 \\ u_3 k_3 - u_5 k_4 &= 0 \\ u'_5 &= 0 \end{aligned}$$

Thus, it is easy to obtain that the coefficients u_1, u_3 and u_5 are given by

$$(4.3) \quad \begin{aligned} u_1 &= \cos \phi \frac{k_2}{k_1} = \text{const } t \\ u_3 &= \cos \phi = \text{const } t \\ u_5 &= \cos \phi \frac{k_3}{k_4} = \text{const } t \end{aligned}$$

Since the coefficients u_1 and u_5 constants, the ratios $\frac{k_2}{k_1}$ and $\frac{k_3}{k_4}$ are constant, respectively. Therefore, if we substitute u_1, u_3 and u_5 in (4.2), we have

$$(4.4) \quad U = \cos \phi \frac{k_2}{k_1} V_1 + \cos \phi V_3 + \cos \phi \frac{k_3}{k_4} V_5.$$

where $\phi \neq k \frac{\pi}{2}$.

Conversely, while the ratios $\frac{k_2}{k_1}$ and $\frac{k_3}{k_4}$ are constant, we can define the vector U . Since the differential of U is $\frac{dU}{ds} = 0$, U is a fixed vector. Furthermore, since $\langle V_3, U \rangle = \cos \phi = \text{const } t$, the curve α become V_3 slant helix. This completes the proof.

Result 4.1. α is a V_3 slant helix if and only if the ratios $\frac{\|V'_2\|}{\|V'_1\|}$ and $\frac{\|V'_4\|}{\|V'_5\|}$ are constant.

Proof: If α is a V_3 slant helix, then from the theorem 4.1 we can write

$$\frac{k_2}{k_1} = \text{const} \tan t$$

and

$$\frac{k_3}{k_4} = \text{const} \tan t.$$

From the Frenet equations, it is easily to see that

$$\frac{\|V_2'\|}{\|V_1'\|} = \sqrt{1 + \left(\frac{k_2}{k_1}\right)^2}$$

and

$$\frac{\|V_4'\|}{\|V_5'\|} = \sqrt{1 + \left(\frac{k_3}{k_4}\right)^2}$$

So we have the ratios

$$\frac{\|V_2'\|}{\|V_1'\|}$$

and

$$\frac{\|V_4'\|}{\|V_5'\|}$$

are constant.

Conversely, if $\frac{\|V_2'\|}{\|V_1'\|} = \text{const} \tan t$ and $\frac{\|V_4'\|}{\|V_5'\|} = \text{const} \tan t$, then the ratios $\frac{k_2}{k_1}$ and $\frac{k_3}{k_4}$ are constant. Thus, from Teorem 4.1 α is V_3 slant helix.

Result 4.2. *If α is a W-curve, then α is V_3 slant helix.*

Proof: If α is a W-curve, then the curvatures k_i , $1 \leq i \leq 4$ are constants. Thus, the ratios $\frac{k_2}{k_1}$ and $\frac{k_3}{k_4}$ constant. This shows that α is V_3 slant helix from Theorem 4.1.

5. V_5 SLANT HELIX (3^{rd} TYPE SLANT HELIX) IN FIVE DIMENSIONAL EUCLIDEAN SPACE E^5

Definition 5.1. A unit speed curve $\alpha : I \subset \mathbb{R} \rightarrow E^5$ is said to be a V_5 slant helix (3^{rd} type slant helix) if the fifth unit Frenet vector field V_5 makes a constant angle Ψ with the unit and fixed direction U .

Theorem 5.1. *Let α be a unit speed curve in E^5 .*

i) α is a V_5 slant helix if and only if the function

$$\frac{1}{k_1^2} \left[\frac{k_4 k_2}{k_3} + f'(s) \right]^2 + [f(s)]^2 + \left(\frac{k_4}{k_3} \right)^2$$

is constant, where $f(s) = \left(\frac{k_4}{k_3} \right)' \frac{1}{k_2}$.

ii) α is a V_5 slant helix if and only if the following equation is satisfied

$$\left[\frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right]' + f k_1 = 0$$

where $f(s) = \left(\frac{k_4}{k_3} \right)' \frac{1}{k_2}$.

Proof: i) From the above definition 5.1, we give the following

$$(5.1) \quad \langle V_5, U \rangle = \cos \psi = \text{const} \tan t$$

If we take the differential of equation (5.1) with respect to s , we obtain

$$\langle -k_4 V_4, U \rangle = 0.$$

Therefore, we may express

$$(5.2) \quad U = u_1 V_1 + u_2 V_2 + u_3 V_3 + u_5 V_5.$$

we know that $u_i = u_i(s)$ and $u_5 = \cos \psi = \text{const} \tan t$

The differentiation (5.2) gives

$$(u'_1 - u_2 k_1) V_1 + (u_1 k_1 + u'_2 - u_3 k_2) V_2 + (u_2 k_2 + u'_3) V_3 + (u_3 k_3 - u_5 k_4) V_4 + u'_5 V_5 = 0$$

and from this equation we find

$$\begin{aligned} u'_1 - u_2 k_1 &= 0 \\ u_1 k_1 + u'_2 - u_3 k_2 &= 0 \\ u_2 k_2 + u'_3 &= 0 \\ u_3 k_3 - u_5 k_4 &= 0 \\ u'_5 &= 0. \end{aligned}$$

Using the above equations, we can form

$$(5.3) \quad \begin{aligned} u_1 &= \frac{\cos \psi}{k_1} \left\{ \frac{k_4 k_2}{k_3} + \left[\left(\frac{k_4}{k_3} \right)' \frac{1}{k_2} \right]' \right\} \\ u_2 &= -\cos \psi \left(\frac{k_4}{k_3} \right)' \frac{1}{k_2} \\ u_3 &= \cos \psi \frac{k_4}{k_3} \\ u_5 &= \cos \psi = \text{const} \tan t. \end{aligned}$$

and

$$(5.4) \quad u'_1 = u_2 k_1.$$

If we define $f = f(s)$ by

$$\left(\frac{k_4}{k_3} \right)' \frac{1}{k_2} = f(s)$$

then the equation (5.3) writes as

$$(5.5) \quad \begin{aligned} u_1 &= \frac{\cos \psi}{k_1} \left(\frac{k_4 k_2}{k_3} + f' \right) \\ u_2 &= -\cos \psi f \\ u_3 &= \cos \psi \frac{k_4}{k_3} \\ u_5 &= \cos \psi = \text{const} \tan t. \end{aligned}$$

and equation (5.4) can be written as

$$(5.6) \quad \left[\frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right]' + f k_1 = 0.$$

Therefore, equation (5.2) takes the following form:

$$(5.7) \quad U = \cos \psi \left(\frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right) V_1 - \cos \psi f V_2 + \cos \psi \frac{k_4}{k_3} V_3 + \cos \psi V_5.$$

Since U is a fixed vector, the following expression

$$(5.8) \quad \frac{1}{k_1^2} \left[\frac{k_2 k_4}{k_3} + f' \right]^2 + f^2 + \left(\frac{k_4}{k_3} \right)^2$$

is obtained as constant.

Conversely, if equation (5.8) holds, then the fixed vector U can be defined as

$$U = \cos \psi \left(\frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right) V_1 - \cos \psi f V_2 + \cos \psi \frac{k_4}{k_3} V_3 + \cos \psi V_5$$

In this case $\frac{dU}{ds} = 0$ and $\langle V_5, U \rangle = \cos \psi = \text{constant}$. It is clear that α is V_5 slant helix.

ii) If α is V_5 slant helix, then from the proof of theorem 5.1 i), we have

$$\left[\frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right]' + f k_1 = 0.$$

Conversely, if the equation (5.6) holds, then the following can be written

$$U = \cos \psi \left(\frac{k_4 k_2}{k_1 k_3} + \frac{f'}{k_1} \right) V_1 - \cos \psi f V_2 + \cos \psi \frac{k_4}{k_3} V_3 + \cos \psi V_5.$$

Since $\frac{dU}{ds} = 0$ and $\langle V_5, U \rangle = \cos \psi = \text{constant}$, α becomes V_5 slant helix.

Result 5.1. *Let α be V_5 slant helix. If the ratio $\frac{k_4}{k_3}$ is constant, then the ratio $\frac{k_2}{k_1}$ is constant. (Namely, α is V_3 slant helix)*

Proof: Suppose that α is V_5 slant helix and the ratio $\frac{k_4}{k_3}$ is constant. So the equation (5.8) becomes constant, that is, the ratio $\frac{k_2}{k_1}$ is found as constant. This means that the curve α is V_3 slant helix from theorem 4.1.

Result 5.2. *If α is V_5 slant helix and $\frac{\|V'_5\|}{\|V'_4\|}$ is constant, then $\frac{\|V'_2\|}{\|V'_1\|}$ is constant. (Namely, α is V_3 slant helix)*

Proof: It is obvious from result 5.1.

Also it can be examined if there are any other helices other than V_1 helix, V_3 slant helix and V_5 slant helix in E^5 .

Theorem 5.2. *Let α be a unit speed curve in E^5 .*

i) *There is no fixed direction making a constant angle with the second Frenet vector V_2 of the curve α .*

ii) *There is no fixed direction making a constant angle with the fourth Frenet vector V_4 of the curve α .*

Proof: Let us assume that the second Frenet vector V_2 of the unit speed curve α in E^5 makes a constant angle with the fixed direction U . So, we can write

$$(5.9) \quad \langle V_2, U \rangle = \cos \beta$$

where β is a constant angle between V_2 and U . Differentiating equation (5.9) with respect to s , we obtain

$$\langle -k_1 V_1 + k_2 V_3, U \rangle = 0.$$

This shows that the vector U is perpendicular to the Frenet vectors V_1 and V_3 , so the following can be written

$$(5.10) \quad U = u_2 V_2 + u_4 V_4 + u_5 V_5, \quad u_i = u_i(s)$$

Differentiating the equation (5.10), we have

$$(-u_2 k_1) V_1 + u'_2 V_2 + (u_2 k_2 - u_4 k_3) V_3 + (u'_4 - u_5 k_4) V_4 + (u_4 k_4 + u'_5) V_5 = 0$$

which leads to the following system

$$(5.11) \quad \begin{aligned} -u_2 k_1 &= 0 \\ u'_2 &= 0 \\ u_2 k_2 - u_4 k_3 &= 0 \\ u'_4 - u_5 k_4 &= 0 \\ u_4 k_4 + u'_5 &= 0 \end{aligned}$$

By taking account of the equations (5.11) we get $u_2 = u_4 = u_5 = 0$ which gives us that $\vec{U} = \vec{0}$.

Moreover, this shows that there is no fixed direction U that makes a constant angle with the Frenet vector V_2 .

ii) The proof is similar to the proof of theorem 5.2.i.

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SAKARYA UNIVERSITY, FACULTY OF EDUCATION, DEPARTMENT OF ELEMENTARY EDUCATION, HENDEK- SAKARYA-TURKEY¹,

E-mail address: mmasal@sakarya.edu.tr, apirdal@sakarya.edu.tr